Department of Mathematics & Philosophy of Engineering

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Course Code:	MPZ3231		

01.

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(a) $f(x) = \frac{x}{x^2-1}$

<u>Local Extremes</u>

$f'(x) = \frac{-(1+x)^2}{(x^2-1)^2} < 0$ for all $x \in \mathbb{R}/\{-1,1\}$

$f'(x)$ and) $f(x)$ both are undefined at $x = \pm 1$

$\therefore$ no local extremes

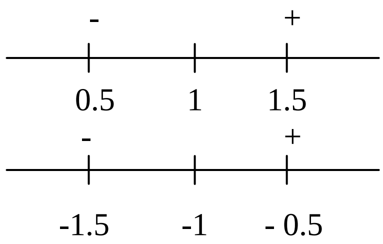
<u>No Inflection Points</u>

<u>Vertical Asymptotes</u>

$f(x)$ is undefined at $x = \pm 1$, vertical asymptotes at $x = \pm 1$.

$\lim_{x\to 1^+} \frac{x}{x^2-1} = \infty$ $\quad$ $\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty$

$\lim_{x\to -1^+} \frac{x}{x^2-1} = \infty$ $\quad$ $\lim_{x\to -1^-} \frac{x}{x^2-1} = -\infty$



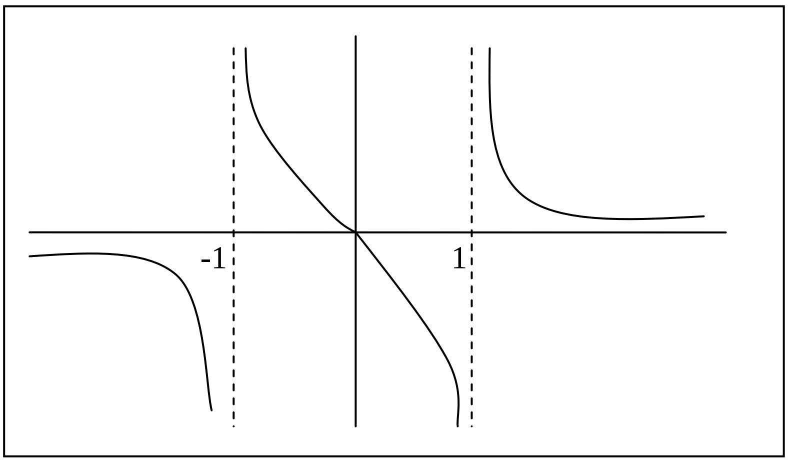
<u>Horizontal Asymptotes</u>

$\lim_{x\to \infty} \frac{x}{x^2-1} = \lim_{x\to \infty} \frac{x}{x^2(1-1/x^2)} = 0$

$\lim_{x\to -\infty} \frac{x}{x^2-1} = \lim_{x\to -\infty} \frac{x}{x^2(1-1/x^2)} = 0$

Horizontal asymptotes $x \to \pm\infty, y \to 0$

<u>At $x = 0$ $y = 0$ $(0,0)$</u>



b)

$$f(x) = \frac{2x+4}{x-1}$$

Local Extremes

$f'(x) = \frac{-6}{(x-1)^2} < 0$ for all $x \in \mathbb{R}/\{1\}$

$f'(x)$ and $f(x)$ both are undefined at $x = 1$

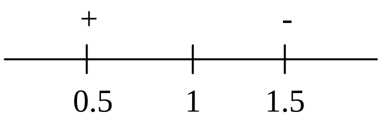
$\therefore$ no local extremes

No Inflection Points

Vertical Asymptotes

$f(x)$ is undefined at $x = 1$, vertical asymptotes at $x = 1$.

$\lim_{x\to 1^+} \frac{2x+4}{x-1} = -\infty$ $\lim_{x\to 1^-} \frac{2x+4}{x-1} = \infty$



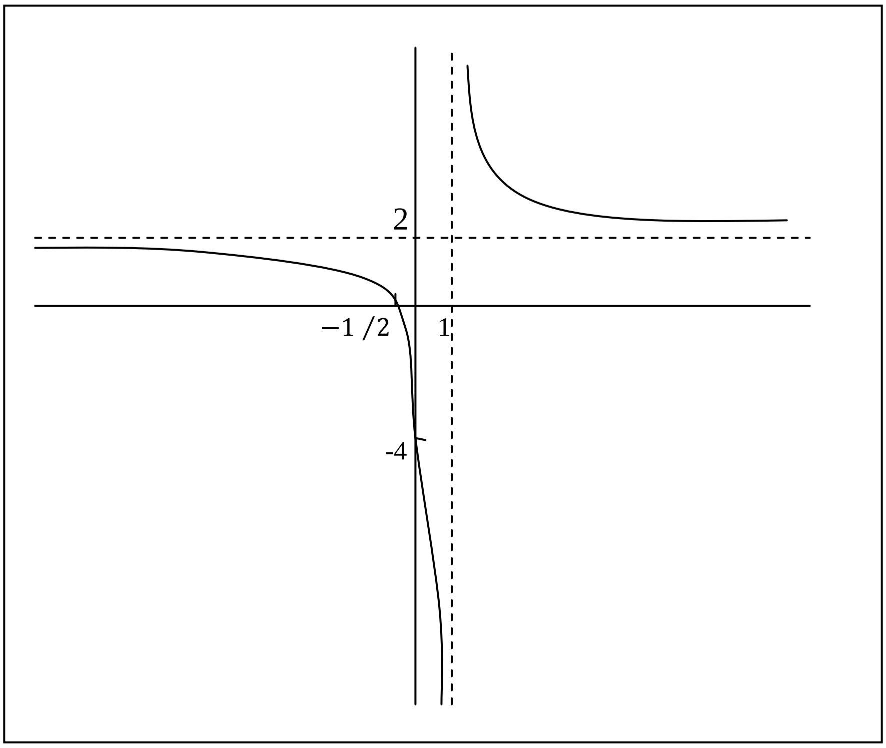
Horizontal Asymptotes

$\lim_{x\to\infty} \frac{2x+4}{x-1} = \lim_{x\to\infty} \frac{x(2+4/x)}{x(1-1/x)} = 2$

$\lim_{x\to-\infty} \frac{2x+4}{x-1} = \lim_{x\to-\infty} \frac{x(2+4/x)}{x(1-1/x)} = 2$

Horizontal asymptotes $x \to \pm\infty, y \to 2$

At $x = 0$ $y = -4$ $(0, -4)$ At $y = 0$ $x = -1/2$ $(-1/2, 0)$



c)

$$f(x) = \frac{x^2 - 4}{x + 1}$$

<u>Local Extremes</u>

$f'(x) = \frac{(x+1)^2+3}{(x+1)^2} = 1 + \frac{3}{(x+1)^2} > 0$ for all $x \in \mathbb{R}/\{-1\}$

$f'(x)$ and $f(x)$ both are undefined at $x = -1$

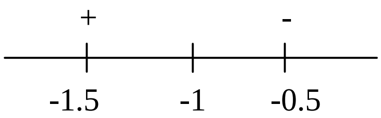
∴ no local extremes

<u>No Inflection Points</u>

<u>Vertical Asymptotes</u>

$f(x)$ is undefined at $x = -1$, vertical asymptotes at $x = -1$.

$\lim_{x \to -1^+} \frac{x^2-4}{x+1} = -\infty$ $\lim_{x \to -1^-} \frac{x^2-4}{x+1} = \infty$



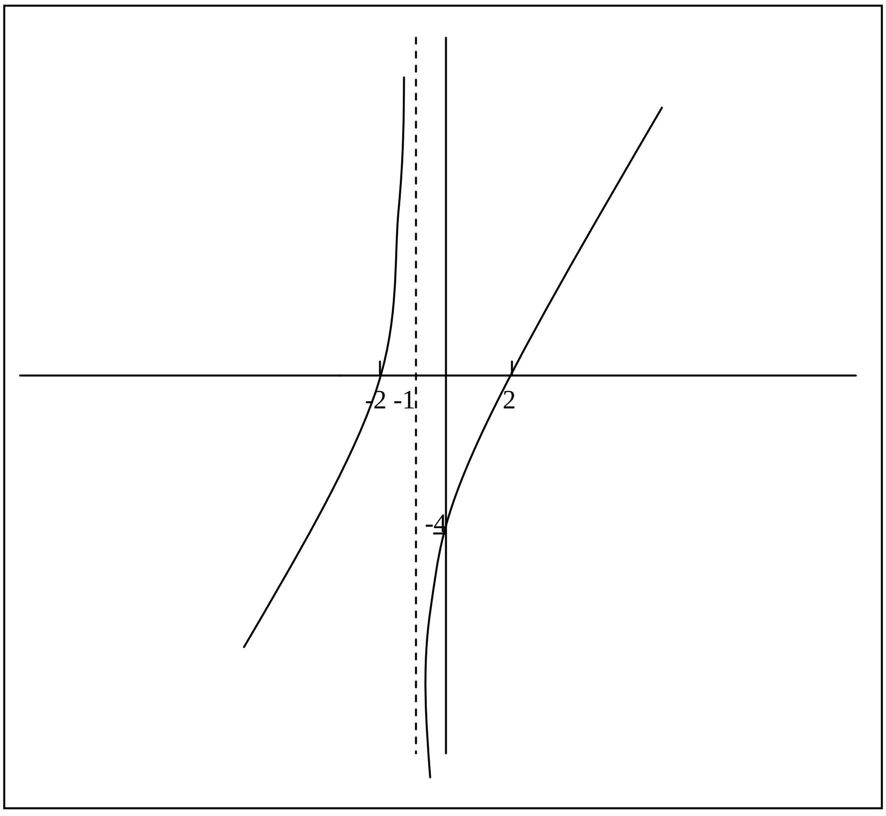
<u>Horizontal Asymptotes</u>

$\lim_{x \to \infty} \frac{x^2-4}{x+1} = \lim_{x \to \infty} \frac{x^2(1-4/x^2)}{x(1+1/x)} = \infty$

$\lim_{x \to -\infty} \frac{x^2-4}{x+1} = \lim_{x \to -\infty} \frac{x^2(1-4/x^2)}{x(1+1/x)} = -\infty$

Horizontal asymptotes :$x \to +\infty, y \to +\infty$ and $x \to -\infty, y \to -\infty$

<u>At</u> $x = 0,\ y = -4,\ (0, -4)$ <u>At</u> $y = 0,\ x^2 - 4 = 0, x = \pm 2, (2,0)$ and $(-2,0)$



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02. (a) $\lim_{x\to\infty^+} \frac{x}{(\ln x)^3 + 2x} = \frac{\infty}{\infty}$ $\lim_{x\to\infty^+} \frac{x}{(\ln x)^3 + 2x} = \lim_{x\to\infty^+} \frac{1}{3(\ln x)^2 \, {}^1\!/_x + 2}$ $\lim_{x\to\infty^+} \frac{3(\ln x)^2}{x} = \frac{\infty}{\infty} \Rightarrow \quad \lim_{x\to\infty^+} \frac{3(\ln x)^2}{x} = \lim_{x\to\infty^+} \frac{6(\ln x)}{1} = 0$ $\therefore \lim_{x\to\infty^+} \frac{x}{(\ln x)^3 + 2x} = \lim_{x\to\infty^+} \frac{1}{3(\ln x)^2 \, {}^1\!/_x + 2} = \frac{1}{2}$	15	100
(b) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \infty - \infty$ $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \frac{0}{0}$ $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x \cos x + \sin x} = \frac{0}{0}$ $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x \cos x + \sin x} = \lim_{x\to 0} \frac{-\sin x}{-x \sin x + \cos x + \cos x} = 0$	17	
(c) $\lim_{x\to 2} \left(\frac{x}{2}\right)^{1/x-2} = 1^\infty$ Let $A = \left(\frac{x}{2}\right)^{1/x-2}, \ln A = \left({}^1\!/_{x-2}\right) \ln\left(\frac{x}{2}\right)$ $A = e^{\left({}^1\!/_{x-2}\right) \ln\left(\frac{x}{2}\right)}$ $\lim_{x\to 2} \left(\frac{x}{2}\right)^{1/x-2} = e^{\left[\lim_{x\to 2}\left[\left({}^1\!/_{x-2}\right) \ln\left(\frac{x}{2}\right)\right]\right]} = e^{\left[\lim_{x\to 2}\left[\ln\left(\frac{x}{2}\right) / (x-2)\right]\right]}$ $\lim_{x\to 2} \left[\ln\left(\frac{x}{2}\right) \middle/ (x-2)\right] = \frac{0}{0}$ $\lim_{x\to 2} \left[\ln\left(\frac{x}{2}\right) \middle/ (x-2)\right] = \lim_{x\to 2} \left[\frac{{}^2\!/_x \times {}^1\!/_2}{1} \right] = {}^1\!/_2$ $\lim_{x\to 2} \left(\frac{x}{2}\right)^{1/x-2} = e^{\left[\lim_{x\to 2}\left[\left({}^1\!/_{x-2}\right) \ln\left(\frac{x}{2}\right)\right]\right]} = e^{\left[\lim_{x\to 2}\left[\ln\left(\frac{x}{2}\right) / (x-2)\right]\right]} = e^{1/2} = \sqrt{e}$	18	

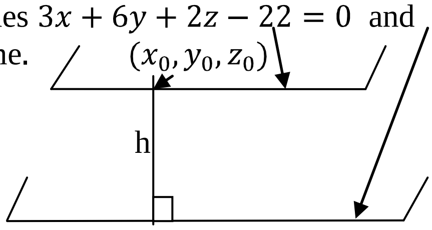
(d) $\lim_{x \to \pi/4} (\tan x - 1) \sec 2x = 0 \times \infty$ $\lim_{x \to \pi/4} (\tan x - 1) \sec 2x = \lim_{x \to \pi/4} \frac{(\tan x - 1)}{\cos 2x} = \frac{0}{0}$ $\lim_{x \to \pi/4} (\tan x - 1) \sec 2x = \lim_{x \to \pi/4} \frac{(\tan x - 1)}{\cos 2x} = \lim_{x \to \pi/4} \frac{sec^2 x}{-2 \sin 2x} = \lim_{x \to \pi/4} \frac{1 + tan^2 x}{-2 \sin 2x} = -1$	17	
(e) $\lim_{x \to 0} \left[\frac{1 + \tan x}{1 + \sin x}\right]^{1/\sin x} = 1^\infty$ $A = \left[\frac{1 + \tan x}{1 + \sin x}\right]^{1/\sin x}$ $\ln A = \frac{1}{\sin x} \ln \left[\frac{1 + \tan x}{1 + \sin x}\right] = \frac{1}{\sin x} [\ln(1 + \tan x) - \ln(1 + \sin x)]$ $A = e^{\frac{1}{\sin x}[\ln(1+\tan x) - \ln(1+\sin x)]}$ $\lim_{x \to 0} \left[\frac{1 + \tan x}{1 + \sin x}\right]^{1/\sin x} = e^{\lim_{x \to 0}\left(\frac{1}{\sin x}[\ln(1+\tan x) - \ln(1+\sin x)]\right)}$ $\lim_{x \to 0} \left(\frac{1}{\sin x} [\ln(1 + \tan x) - \ln(1 + \sin x)]\right) = \frac{0}{0}$ $\lim_{x \to 0} \left(\frac{1}{\sin x} [\ln(1 + \tan x) - \ln(1 + \sin x)]\right) = \lim_{x \to 0} \left(\frac{\left[\frac{(1 + \tan^2 x)}{(1 + \tan x)} - \frac{\cos x}{(1 + \sin x)}\right]}{\cos x}\right) = 0$ $\lim_{x \to 0} \left[\frac{1 + \tan x}{1 + \sin x}\right]^{1/\sin x} = e^0 = 1$	18	
(f) $\lim_{x \to 0} \frac{\ln(1 + x) - \sin x}{x \sin x} = \frac{0}{0}$ $\lim_{x \to 0} \frac{\ln(1 + x) - \sin x}{x \sin x} = \lim_{x \to 0} \frac{\frac{1}{1 + x} - \cos x}{x \cos x + \sin x} = \frac{0}{0}$ $\lim_{x \to 0} \frac{\ln(1 + x) - \sin x}{x \sin x} = \lim_{x \to 0} \frac{\frac{1}{1 + x} - \cos x}{x \cos x + \sin x} = \lim_{x \to 0} \frac{\frac{-1}{(1 + x)^2} + \sin x}{\cos x + \cos x + x \ \sin x} = -\frac{1}{2}$	15	

03. (a) $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[\left(e^{x^2-y^2}\right)^2 \cdot \sin(xy^2)\right] = \frac{\partial}{\partial x}\left[\left(e^{2(x^2-y^2)}\right) \cdot \sin(xy^2)\right]$ $= e^{2(x^2-y^2)} \cdot \cos(xy^2) \cdot y^2 + e^{2(x^2-y^2)} \cdot 2 \cdot 2x \cdot \sin(xy^2)$ $= e^{2(x^2-y^2)}(\cos(xy^2) \cdot y^2 + 4x \cdot \sin(xy^2))$ $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[\left(e^{x^2-y^2}\right)^2 \cdot \sin(xy^2)\right] = \frac{\partial}{\partial y}\left[\left(e^{2(x^2-y^2)}\right) \cdot \sin(xy^2)\right]$ $= e^{2(x^2-y^2)} \cdot \cos(xy^2) \cdot 2xy + e^{2(x^2-y^2)} \cdot 2 \cdot -2y \cdot \sin(xy^2)$ $= 2e^{2(x^2-y^2)}y(x\cos(xy^2) - 2\sin(xy^2))$	20	100
(b) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$ $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(e^v)$ $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} + e^v)\frac{\partial z}{\partial y}$ $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$	30	
(c) $v = (11 = 2xy + y^2)^{-1/2}$ $\frac{\partial v}{\partial y} = -{}^{1}/_{2}\,(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = v^3(x - y)$ $\frac{\partial v}{\partial x} = -{}^{1}/_{2}\,(1 - 2xy + y^2)^{-3/2}(0 - 2y) = yv^3$ $y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} = (yx - y^2)v^3 - xyv^3 = -y^2v^3$	20	

(d) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$ $\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \sin^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta$ $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}-\mathrm{r}\sin\theta + \frac{\partial u}{\partial y}\ \mathrm{r}\cos\ \theta$ $\left(\frac{1}{r^2}\right)\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta$ $\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 (\sin^2\theta + \cos^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (\sin^2\theta + \cos^2\theta)$ $\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$	30	

(4) 100

(i) planes $3x + 6y + 2z - 22 = 0$ and $3x + 6y + 2z - 27$ are parallel since the coefficients are same. 20

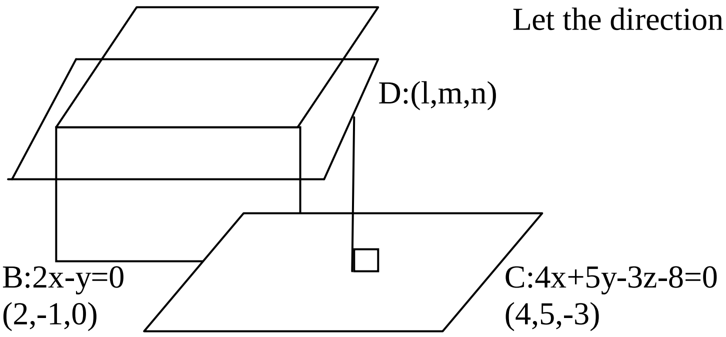


$$h = \left|\frac{3x_0+6y_0+2z_0-27}{\sqrt{6^2+3^2+2^2}}\right|$$

$$h = \left|\frac{22-27}{\sqrt{6^2+3^2+2^2}}\right| = \frac{5}{7}$$

(ii) A:3z-y=0 (0,-1,3)

Let the direction ratios of the plane D be (l,m,n) 50



Line of the intersection of the two planes A and B be L and L is given by:
$3z - y = 2x - y = 0$
L:$3z = 2x = y$
Direction ratios of L: $1/2,1,1/3$ (direction ratios of L and D are perpendicular)
Then $l/2 + m + n/3 = 0$ --------------------(1)
Direction ratios of C: 4,5,-3 (direction ratios of C and D are perpendicular)
Then $4l + 5m - 3n = 0$ ----------------------(2)

From (1) and (2) $l = -28m/17$ $n = -9m/17$
Direction ratios of the plane are $28, -17, 9$
Let (x,y,z) be a point on the plane D and since (0,0,0) is a point on the line L, thus point on the plane D. equation of the plane D is : $28x - 17y + 9z = 0$

(iii) 30

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & \alpha \end{pmatrix} \qquad C = \left(\begin{array}{ccc|c} 1 & 2 & 1 & \beta \\ 3 & -5 & 3 & 1 \\ 2 & 7 & \alpha & 8 \end{array}\right)$$

$$C = \left(\begin{array}{ccc|c} 1 & 2 & 1 & \beta \\ 3 & -5 & 3 & 1 \\ 2 & 7 & \alpha & 8 \end{array}\right) \xrightarrow[\substack{r2 \to r2+-3r1 \\ r3 \to r3+-2r1}]{} \left(\begin{array}{ccc|c} 1 & 2 & 1 & \beta \\ 0 & -11 & 0 & 1-3\beta \\ 0 & 3 & \alpha-2 & 8-2\beta \end{array}\right) \xrightarrow[r2 \to r2/-11]{}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & \beta \\ 0 & 1 & 0 & 1-3\beta/-11 \\ 0 & 3 & \alpha-2 & 8-2\beta \end{array}\right) \xrightarrow[R3 \to r3+-3r2]{} \left(\begin{array}{ccc|c} 1 & 2 & 1 & \beta \\ 0 & 1 & 0 & 3\beta-1/11 \\ 0 & 0 & \alpha-2 & 91-31\beta \end{array}\right)$$

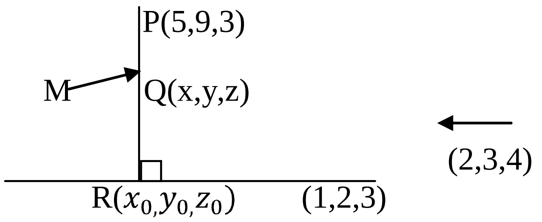
(a) Rank A = rank C = 3
$\alpha - 2 \neq 0 \Longrightarrow \alpha \neq 2, \beta \in \mathbb{R}$

(b) Rank A = rank C = $r < 3$
$\alpha - 2 = 0$ and $91 - 31\beta = 0 \Longrightarrow \alpha = 2$ and $\beta = {}^{91}/_{31}$

(c) Rank A $\neq$ rank C
$\alpha - 2 = 0$ and $91 - 31\beta \neq 0 \Longrightarrow \alpha = 2$ and $\beta \neq {}^{91}/_{31}$

05. (i) 20 100

(a)



L: $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$

Let t be a parameter then on the line L
$x = 2t + 1 \quad x_0 = 2t + 1$
$y = 3t + 2 \quad y_0 = 3t + 2$
$z = 4t + 3 \quad z_0 = 4t + 3$

Since P and R lie on the line M, direction coefficients of M are

$$2t + 1 - 5 : 3t + 2 - 9 : 4t + 3 - 3$$
$$2t - 4 : 3t - 7 : 4t$$

Direction coefficients of L and M are perpendicular

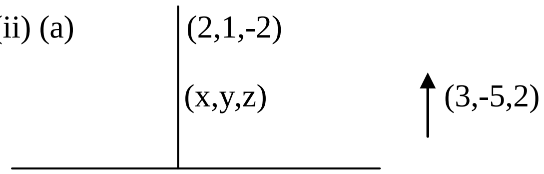
$$2(2t - 4) + 3(3t - 7) + 3 \times 4t = 0$$
$$t = 1$$

Direction ratios of M: -2,-4,4 = 1,2,-2
Equation of the line M: $\frac{x-5}{1} = \frac{y-9}{2} = \frac{z-3}{-2}$

Point R: (3,5,7)

(b) lines to be perpendicular 5
$-3 \times 3k + 2k \times 1 + 2 \times -5 = 0$
$k = -10/7$

(ii) (a)



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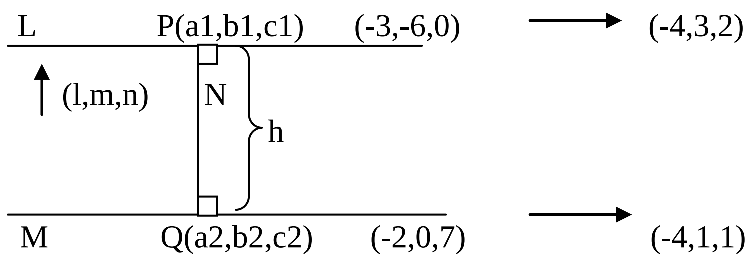
Equation of the line

$$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$$

(b)



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Let h be the shortest distance between the two lines.

Since lines L and N are perpendicular
-4l+3m+2n=0 -------------(1)
Since lines M and N are perpendicular
-4l+m+n=0 ---------------(2)

(1)-(2) m=-(1/2)n
(1)-3(2) l=(1/8)n

Direction ratios of N l:m:n = 1: (-1/4) : 8

Since P is on L since Q is on M where t and T are parameters.

a1=-4t-3	a2=-4T-2
b1=3t-6	b2=T
c1=2t	c2=T+7

then $\frac{-4t-3+4T+2}{1} = \frac{3t-6-T}{-4} = \frac{t-T-7}{8}$ $\because \frac{a1-a2}{1} = \frac{b1-b2}{-4} = \frac{c1-c2}{8}$ on the line N

then t=35/9 T=109/27

a1=-167/9	a2=-490/27
b1=51/9	b2=109/27
c1=70/9	c2=298/27

$$h = \sqrt{\left(\frac{-167}{9} - \frac{490}{27}\right)^2 + \left(\frac{51}{9} - \frac{109}{27}\right)^2 + \left(\frac{70}{9} - \frac{298}{27}\right)^2} = \sqrt{\frac{121}{9}} = \frac{11}{9}$$

equation of N

$$\frac{x + \frac{167}{9}}{1} = \frac{y - \frac{51}{9}}{-4} = \frac{z - \frac{70}{9}}{8}$$

(iii) let the tangent be $y = mx + c$

Since it is parallel to $x - y + 5 = 0, m = 1$ 15

Consider

$$4x^2 - 2(mx + c)^2 - 3x(mx + c) + 2x - 3(mx + c) - 10 = 0$$
$$4x^2 - 2(x + c)^2 - 3x(x + c) + 2x - 3(x + c) - 10 = 0$$
$$x^2 + x(7c + 1) + (2c^2 + 3c + 10) = 0$$

Condition for the tangency, $\Delta = 0$

$$(7c + 1)^2 - 4(2c^2 + 3c + 10) = 0$$

$c = 39/41$ or $c = -1$
equations of the tangent lines are
$y = x + 39/41$ or $y = x - 1$

(iv) let the tangent be $y = mx + c$
Since it is perpendicular to $x - 2y = 7,\ m = -2$

Consider $x(mx + c) = 2$ 15

$$x(-2x + c) = 2$$
$$(-2x^2 + cx - 2) = 0$$

Condition for the tangency, $\Delta = 0$

$$c^2 - 4(-2)(-2) = 0$$

$$c^2 - 16 = 0$$ $$c = \pm 4$$ Equations of the tangent lines are $y = -2x + 4$ or $y = -2x - 4$		